Discussion 11 Worksheet

Lagrange Multipliers

Date: 10/4/2021

MATH 53 Multivariable Calculus

1 Lagrange Multipliers

1. Find the extreme values of the function $f(x, y) = 2x + y + 2z$ subject to the constraint that $x^2 + y^2 + z^2 = 1$.

2. Find the extreme values of the function $f(x, y) = y^2 e^x$ on the domain $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.

3. Use Lagrange multipliers to find the closest point(s) on the parabola $y = x^2$ to the point $(0, 1)$. How could one solve this problem without using any multivariate calculus?

4. You have 24 square inches of cardboard and want to build a box (in the shape of a rectangular prism). Show that a 2" × 2" × 2" cube encloses the largest volume.

5. Find the largest possible volume of a rectangular prism with edges parallel to the coordinate axes and all vertices lying on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

(where $a, b, c > 0$.)

6. Use Lagrange multipliers to find the closest points to the origin on the hyperbola $xy = 1$.

2 Lagrange multipliers with two constraints

1. Maximize and minimize $3x - y - 3z$ subject to $x + y - z = 1$ and $x^2 + 2z^2 = 1$.

2. Maximize and minimize z subject to $x^2 + y^2 = z^2$ and $x + y + z = 24$.

3 Challenge

1. Using the method of Lagrange multipliers, prove the following inequality: if $x_1, \ldots, x_n$ are positive real numbers, then

$$\frac{n}{1/x_1 + \ldots + 1/x_n} \leq \sqrt[n]{x_1 \ldots x_n}$$

with equality if and only if $x_1 = x_2 = \ldots = x_n$. The lefthand side is called the *harmonic mean* of the numbers $x_1, \ldots, x_n$ and the righthand side is called their *geometric mean*.

2. As in problem 1.4., find the dimensions of the box enclosing the largest volume if the box has no top. Hint: try making a substitution before using Lagrange multipliers.

3. If $x_1, \ldots, x_n$ are real numbers, prove that

$$\left(\sum_{i=1}^{n} x_i\right)^2 \leq n\left(\sum_{i=1}^{n} x_i^2\right).$$

4 True/False

Supply convincing reasoning for your answer.

(a) T F Any continuous function on the domain $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ will attain a maximum.

(b) T F If $xye^x = \lambda y$ and $xye^x = \lambda x$, then we can conclude that $x = y$.

(c) T F If $f(x,y)$ is differentiable and attains a maximum at (a,b) in the region $\{(x,y) \in \mathbb{R}^2 : |x| + |y| \leq 1\}$, then $f_x(a,b) = f_y(a,b) = 0$.

(d) T F It is possible that a function $f(x,y)$ can have no extrema along a level curve $g(x,y) = 0$.

Note: These problems are taken from the worksheets for Math 53 in the Spring of 2021 with Prof. Stankova.